

Ex. Show that the set  $S = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$  is a basis for  $M_{2 \times 2}$

a)  $S$  spans  $M_{2 \times 2}$

$$\begin{pmatrix} 4 & 6 \\ -2 & -3 \end{pmatrix} = 4 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + 6 \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + (-2) \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + (-3) \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$b) C_1 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + C_2 \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + C_3 \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + C_4 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$C_1 = C_2 = C_3 = C_4 = 0 \quad \text{Linearly independent}$$

$\therefore S$  is a basis for  $M_{2 \times 2}$

### 3. The dimension of a vector space

Def. If a vector space  $V$  has a basis consisting of  $n$  vectors, then the number  $n$  is called the dimension of  $V$  denoted by  $\dim(V) = n$

$$\text{eg. } \dim(M_{2 \times 2}) = 4 \quad (4 \text{ vectors in basis of } M_{2 \times 2})$$

$$\text{eg. } \dim(\mathbb{R}^3) = 3 \quad (3 \text{ vectors in basis of } \mathbb{R}^3)$$

$$\text{eg. } \dim(p_n) = n+1$$

$$\text{eg. } \dim(M_{m \times n}) = m \times n$$

$$\text{eg. } \dim(\mathbb{R}^n) = n$$

} General

### The rank of a matrix

Row and Column space of a matrix

Let  $A$  be an  $m \times n$  matrix

① The row space of  $A$  is the space spanned by the row vectors of  $A$

② The column space of  $A$  is the space spanned by the column vectors of  $A$

$$\text{Ex } A = \begin{pmatrix} 3 & 2 & 0 \\ 8 & 7 & -4 \end{pmatrix}$$

$$\text{Row space } \{ (3, 2, 0) \ (8, 7, -4) \} \in \mathbb{R}^3$$

$$\text{Column space } \{ (3, 8)^T \ (2, 7)^T \ (0, -4)^T \} \in \mathbb{R}^2$$

$$\text{Row vector } \{ (3, 2, 0) \ (8, 7, -4) \}$$

$$\text{Column vector } \left\{ \begin{pmatrix} 3 \\ 8 \end{pmatrix} \begin{pmatrix} 2 \\ 7 \end{pmatrix} \begin{pmatrix} 0 \\ -4 \end{pmatrix} \right\}$$



$$\text{Ex. } A = \begin{pmatrix} 10 & 0 & 5 & 4 \\ 0 & 2 & -7 & 1 \\ 3 & 1 & 2 & 1 \end{pmatrix}$$

Row space  $\{(10, 0, 5, 4) (0, 2, -7, 1) (3, 1, 2, 1)\} \in \mathbb{R}^4$

Column space  $\{(10, 0, 3)^T (0, 2, 1)^T (5, -7, 2)^T (4, 1, 1)^T\} \in \mathbb{R}^3$

How to find the basis for a row space of a matrix?

Theorem: If a matrix  $A$  is row equivalent to another matrix  $B$  in row echelon form, then the non-zero rows of  $B$  form a basis for the row space of  $A$ .

Ex. Find the basis for the row space of

$$A = \begin{pmatrix} 1 & 3 & 1 & 3 \\ 0 & 1 & 1 & 0 \\ -3 & 0 & 6 & -1 \\ 3 & 4 & -2 & 1 \\ 2 & 0 & -4 & -2 \end{pmatrix} \xrightarrow[\text{form}]{\text{echelon}} \begin{pmatrix} 1 & 3 & 1 & 3 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow \text{Basis } \left\{ \begin{pmatrix} 1 & 3 & 1 & 3 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix} \right\}$$

$\text{rank}(A) = 3$

Def. The rank of a matrix  $A$  is the dimension of the row space of the matrix  $A$  and is denoted by  $\text{rank}(A)$ .

Ex. Find the rank of  $A$

$$A = \begin{pmatrix} 1 & -2 & 0 & 1 \\ 2 & 1 & 5 & -3 \\ 0 & 1 & 3 & 5 \end{pmatrix} \xrightarrow{R_2 - 2R_1} \begin{pmatrix} 1 & -2 & 0 & 1 \\ 0 & 5 & 5 & -5 \\ 0 & 1 & 3 & 5 \end{pmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{pmatrix} 1 & -2 & 0 & 1 \\ 0 & 1 & 3 & 5 \\ 0 & 5 & 5 & -5 \end{pmatrix}$$

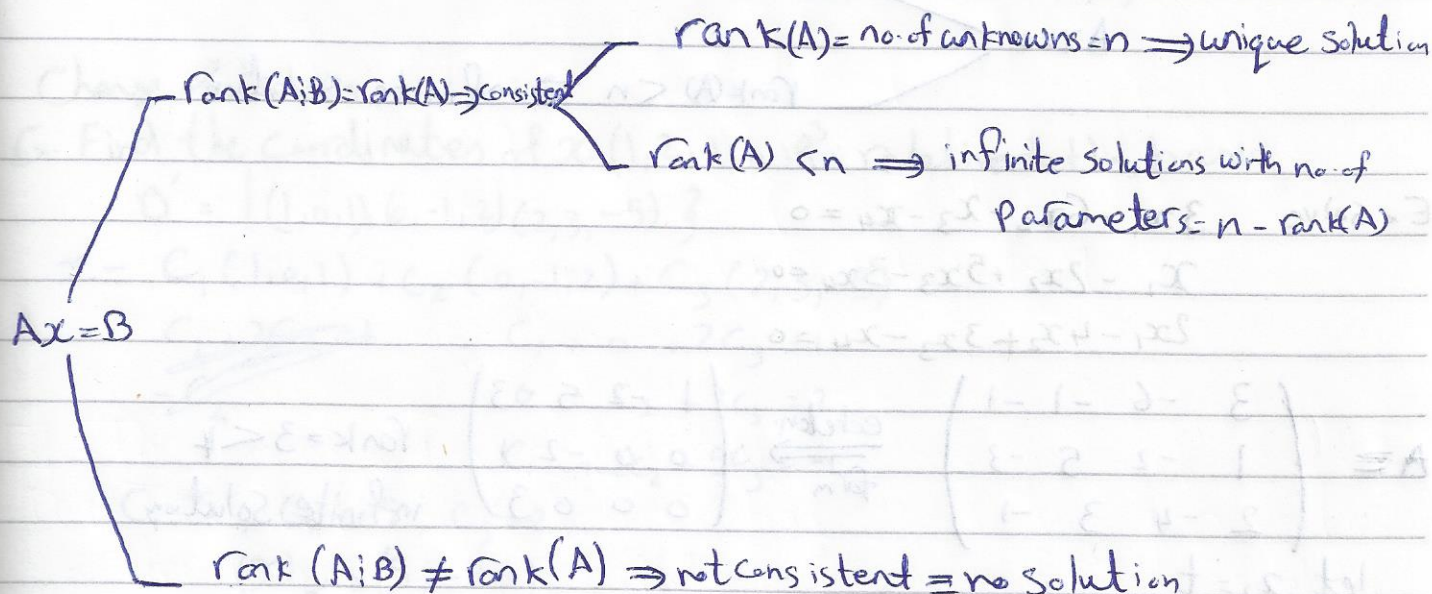
$$\xrightarrow{R_3 - 5R_2} \begin{pmatrix} 1 & -2 & 0 & 1 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & -10 & -30 \end{pmatrix} \Rightarrow \text{Basis } \left\{ \begin{pmatrix} 1 & -2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 3 & 5 \end{pmatrix} \begin{pmatrix} 0 & 0 & -10 & -30 \end{pmatrix} \right\}$$

$\text{rank}(A) = 3$



## Rank of matrix and system of linear equations

### □ System of non-homogeneous equations ( $AX=B$ )



Solve:  $x_1 - 2x_3 + x_4 = 5$

$3x_1 + x_2 - 5x_3 = 8$

$x_1 + 2x_2 - 5x_4 = -9$

$(A:B) = \begin{pmatrix} 1 & 0 & -2 & 1 & 5 \\ 3 & 1 & -5 & 0 & 8 \\ 1 & 2 & 0 & -5 & -9 \end{pmatrix}$

$(A:B) = \begin{pmatrix} 1 & 0 & -2 & 1 & 5 \\ 3 & 1 & -5 & 0 & 8 \\ 1 & 2 & 0 & -5 & -9 \end{pmatrix} \xrightarrow[R_3 - R_1]{R_2 - 3R_1} \begin{pmatrix} 1 & 0 & -2 & 1 & 5 \\ 0 & 1 & 1 & -3 & -7 \\ 0 & 2 & 2 & -6 & -14 \end{pmatrix} \xrightarrow{R_3 - 2R_2}$

$\begin{pmatrix} 1 & 0 & -2 & 1 & 5 \\ 0 & 1 & 1 & -3 & -7 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

$\text{Rank}(A) = 2 \Rightarrow \text{consistent}$   
 $\text{Rank}(A:B) = 2$

$\text{Rank}(A) = 2 < 4 \Rightarrow \text{infinite no. of solutions}$

Let  $x_3 = s, x_4 = t$

$x_2 = -7 - s + 3t$

$x_1 = 5 + 2s - t$

$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 5 + 2s - t \\ -7 - s + 3t \\ s \\ t \end{pmatrix}$



## 2 System of homogeneous equations ( $Ax=0$ )

$$Ax=0 \begin{cases} \text{rank}(A) = n (\text{no. unknowns}) \Rightarrow \text{trivial solution} \\ \text{rank}(A) < n \Rightarrow \text{infinite solutions} \end{cases}$$

Ex. Solve  $3x_1 - 6x_2 - x_3 - x_4 = 0$

$$x_1 - 2x_2 + 5x_3 - 3x_4 = 0$$

$$2x_1 - 4x_2 + 3x_3 - x_4 = 0$$

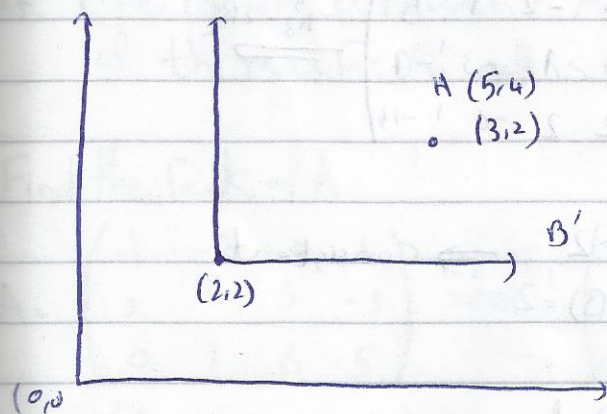
$$A \equiv \begin{pmatrix} 3 & -6 & -1 & -1 \\ 1 & -2 & 5 & -3 \\ 2 & -4 & 3 & -1 \end{pmatrix} \xrightarrow[\text{form}]{\text{echelon}} \begin{pmatrix} 1 & -2 & 5 & -3 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & 3 \end{pmatrix} \quad \begin{matrix} \text{rank} = 3 < 4 \\ \text{infinite solutions} \end{matrix}$$

let  $x_2 = t$

$$x_1 = 2t, \quad x_3 = 0, \quad x_4 = 0$$

## Coordinate System and change of basis

eg.  $\mathbb{R}^2 \Rightarrow$  basis  $B = \{(0,1), (1,0)\}$



$$H = (3,2)_{B'}$$

Def: Let  $B: \{v_1, v_2, \dots, v_n\}$  to be basis for a vector space  $V$  and let  $x$  be a vector in  $V$  such that  $x = c_1 v_1 + c_2 v_2 + \dots + c_n v_n$ , then the scalars  $c_1, c_2, \dots, c_n$  are called the coordinates of  $x$ .



Ex.  $x = (5, 8, 4, -1) \in \mathbb{R}^4$

basis of  $\mathbb{R}^4$   $\{(1, 0, 0, 0) (0, 1, 0, 0) (0, 0, 1, 0) (0, 0, 0, 1)\}$

$$x = 5(1, 0, 0, 0) + 8(0, 1, 0, 0) + 4(0, 0, 1, 0) - 1(0, 0, 0, 1)$$

Change of basis:

Ex. Find the coordinates of  $x = (1, 2, -1)$  in  $\mathbb{R}^3$  relative to the basis

$$B' = \{(1, 0, 1), (0, -1, 2), (2, 3, -5)\}$$

$$x = C_1(1, 0, 1) + C_2(0, -1, 2) + C_3(2, 3, -5)$$

$$\begin{aligned} C_1 + 2C_3 &= 1 \\ -C_2 &= 2 \\ C_1 + 2C_2 - 5C_3 &= -1 \end{aligned}$$

$$C_1 + 0 + 2C_3 = 1$$

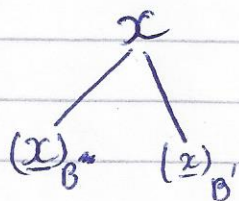
$$0 - C_2 + 3C_3 = 2$$

$$C_1 + 2C_2 - 5C_3 = -1$$

$$C_1 = 5, C_2 = -8, C_3 = 2$$

$$(x)_{B'} = (5, -8, 2)$$

How to change basis



$$(x)_{B'} = P^{-1}(x)_B$$

$P^{-1}$  = transition matrix

$(x)_B$  = the general coordinates

To find the transition matrix, we use row operations to make the following form  $(B' | B) \Rightarrow (I_n | P^{-1})$

Ex)  $(x)_B = (1, 2, -1)$

$$B = \{(1, 0, 0) (0, 1, 0) (0, 0, 1)\}$$

$$B' = \{(1, 0, 1) (0, -1, 2) (2, 3, -5)\}$$

Find  $(x)_{B'}$

$$\left( \begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & -1 & 2 & 0 & 1 & 0 \\ 2 & 3 & -5 & 0 & 0 & 1 \end{array} \right) \Rightarrow \left( \begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 4 & 2 \\ 0 & -1 & 0 & 3 & -7 & -3 \\ 0 & 0 & 1 & 1 & -2 & -1 \end{array} \right)$$

$$(x)_{B'} = P^{-1}(x)_B = \begin{pmatrix} -1 & 4 & 2 \\ 3 & -7 & -3 \\ 1 & -2 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 5 \\ -8 \\ -2 \end{pmatrix}$$